# GEOMETRY FINAL EXAM (April 2026)

Answer all the questions. Total 50 marks.

(You may use any theorem proved in class, but you must state it clearly.)

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**Note.** Here, we assume $V$ to be a finite dimensional vector space with $P(V)$ to be projective space over $V$. Also, we define $P_n(\mathbb{R}) := P(\mathbb{R}^{n+1})$ and $P_n(\mathbb{C}) := P(\mathbb{C}^{n+1})$.

(1) [5 points] Let $\mathsf{PGL}(V)$ be the projective group over the vector space $V$ with $\dim V = n+1$. Let $\varphi \in \mathsf{PGL}(V)$. If there exists a projective frame $(m_0, m_1, \ldots, m_{n+1})$ in $P(V)$ such that $\varphi(m_i) = m_i$, for every $i = 0, 1, \ldots, n+1$, then show that $\varphi$ is the identity transformation.

(2) [5 points] Prove that every projective transformation $\varphi : P_n(\mathbb{C}) \longrightarrow P_n(\mathbb{C})$ has at least one fixed point, for each $n \geq 1$.

(3) [6 points] Let $L$ be a projective line (i.e., a one dimensional projective space). A projective transformation $\varphi : L \longrightarrow L$ (i.e., $\varphi$ is a *homography*) is an *involution* if $\varphi \circ \varphi = Id$ ($\varphi \neq Id$). Prove that for any two disjoint pairs of distinct points $(A, A')$ and $(B, B')$ on $L$, there exists a unique involution $h : L \longrightarrow L$ such that $h(A) = A'$ and $h(B) = B'$.

(4) [3+6 points] Let $\mathcal{A}$ be a finite dimensional affine space.

- ($a$) Define the projective completion of $\mathcal{A}$.
- ($b$) Let $\widehat{\mathcal{A}}$ be the projective completion of $\mathcal{A}$. Let $f : \mathcal{A} \longrightarrow \mathcal{A}$ be an affine transformation, then show that $f$ extends to a unique projective transformation $\widehat{f} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}}$.

(5) [6 points] Let $\mathcal{A}$ be a real affine space of dimension $n \geq 2$, and $\widehat{\mathcal{A}}$ be its projective completion. Let $H_\infty$ denotes the hyperplane at infinity. Let $L \subset \widehat{\mathcal{A}}$ be a projective line containing two distinct points $A, B \in \mathcal{A}$. Suppose $P_\infty \in L \cap H_\infty$ and $C \in L \cap \mathcal{A}$. Prove that $C$ is the midpoint of the line segment $AB$ if and only if the cross-ratio satisfies

$$[A, B, C, P_\infty] = -1$$

(6) [6 points] Let $H_1$ and $H_2$ be two distinct hyperplanes in a projective space $P(V)$. Prove that the set of hyperplanes containing $H_1 \cap H_2$ is a projective line in the dual space of $P(V)$.

(7) [4 points] Let $\varphi$ be an element of the circular group $G$ (i.e., $G$ be the group of transformation of $P_1(\mathbb{C})$ generated by the homographies and the symmetry). Assume that $\varphi(\infty) = \infty$. Does $\varphi$ preserve oriented angles?

(8) [3+6 points]

- ($a$) State the affine version of Pappus' theorem.
- ($b$) State and prove the projective version of Pappus' theorem assuming the affine version.